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Young Children's Ability to Use the Balance Strategy to Solve for Unknowns

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This article examines students' ability to use the balance model to solve for unknowns. A teaching experiment was conducted in four Year 3 classrooms. This experiment focused on exploring the application of the balance model as an analogue for representing equations and solving for unknowns. The teaching experiment promoted a shift by students towards viewing addition and subtraction equations in terms of equivalence, where the situation is viewed in a multi-directional way (i.e., balance). Initially the lessons were trialed by the researchers in two classrooms. The lessons were then implemented in a further two classrooms by two classroom teachers in conjunction with the researchers. Two weeks after the conclusion of the teaching experiment, a one-on-one interview was conducted with a random sample comprising twenty students with an average age of eight years. The interviews indicated that while the balance model did assist students reach solutions for finding unknowns, for some students further explicit teaching was required.

Introduction

In spite of the wealth of research that has occurred in the algebraic domain (e.g., Boulton-Lewis, Cooper, Atweh, Pillay & Wills, 1998; Linchevski & Herscovics, 1996; Rojano & Sutherland, 1994), students are still experiencing many difficulties. Recent research has turned to investigate how young children embed algebraic reasoning *in* arithmetic reasoning. This is a shift from the traditional approach of algebraic reasoning that occurs after the development of arithmetic reasoning to algebraic reasoning that occurs in conjunction with arithmetic reasoning. One of the most pressing factors for algebraic reform is the ability of elementary teachers to *algebrafy* arithmetic (Kaput & Blanton, 2001), that is, to develop in their students the arithmetic underpinnings of algebra (Warren & Cooper, 2001), and extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). As was argued by Carpenter and Levi (2000), the artificial separation of arithmetic and algebra "deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years" (p. 1). The research reported here begins to explore young children's understanding of equivalence, an important aspect of algebraic thinking, and represents the first stage of a three year longitudinal study.

Equivalence

Past research has shown that students possess a narrow and restricted knowledge of arithmetic at an early age. This is believed to impede the later development of algebraic thinking (e.g., Carpenter, Franke, & Levi, 2003). An example of this restricted knowledge is young children's understanding of the *equal sign*. Many interpret it as meaning, "Here comes the answer" (Carpenter et al., 2003; Warren, 2002). Another example is young children's limited understanding of the commutative law. For example, when 73 Year 3 students were asked if 2 + 3 = 3 + 2 and 2 – 3 = 3 – 2 were both true, only 25% indicated the first equation was true and the second false (Warren, 2001). The results from that study indicated that teaching materials themselves can act as cognitive obstacles to abstracting the underlying structure of arithmetic. Many of these students had very limited experience with equations written in the horizontal format. The position of the equal sign also caused difficulties, and this finding supports past research that many children in elementary grades generally think that the equal sign means that they should carry out the calculation that precedes it and that the number following the equal sign is the answer to the calculation (Saenz-Ludlow & Walgamuth, 1998). It seems that, as Malara and Navarra (2003) argued, classroom activities in the early years focus on mathematical products rather than on mathematical processes, and this results in limited cognisance and misconceptions. Once these misconceptions exist they are very hard to change (Carpenter et al., 2003) and become even more entrenched as children progress through schooling (e.g., Warren, 2003).

This article reports on the results of a teaching experiment aimed at broadening students' understanding of the equal sign to include equal as representing equivalence. In mathematics the use of the equal sign appears to fall into four main categories. These are (a) the result of a sum (e.g., 3 + 4 = 7), (b) quantitative sameness (e.g., 1 + 3 = 2 + 2), (c) a statement that something is true for all values of the variable (e.g., x + y = y + x), and (d) a statement that assigns a value to a new variable (e.g., x + y = z) (Freudenthal, 1983). With regard to quantitative sameness, "equals" means that both sides of an equation are the same and that information can be from either direction in a symmetrical fashion (Kieran & Chalouh, 1992). Most students do not have this understanding; rather they have a persistent idea that the equal sign is either a *syntactic indicator* (i.e., a symbol indicating where the answer should be written) or an *operator sign* (i.e., a stimulus to action or "to do something") (Behr, Erlwanger & Nicols, 1980; Carpenter & Levi, 2000; Saenz-Ludlow & Walgamuth, 1998; Warren, 2001). For example, when considering 3 + 4 = 6 + 1, students who believe that the equal sign is a syntactic indicator would suggest that this statement is incorrect as 3 + 4 = 7 not 6: the answer occurs after the equal sign. By contrast students who view equals as an operator sign would suggest that the statement is incorrect because 3 + 4 = 7 + 1 = 8 or 3 + 4 = 7 + 6 = 13 + 1 = 14. These incorrect understandings of equals appear to continue into secondary and tertiary education (Baroody &

Ginsburg, 1983; Steinberg, Sleeman, & Ktoriza, 1991) and seem to affect mathematics learning at these levels. Helping students go from equals as a syntactic indicator or an operator sign to quantitative sameness is not an easy task (Saenz-Ludlow & Walgamuth, 1998). The particular focus of this research was on endeavouring to develop understandings of equal as quantitative sameness by exploring the language used to describe equivalent situations, developing equations as the act of stating the identity of two quantities or expressions, and using models and symbolic representations to assist this exploration and development.

The model chosen for this research was the balance model, with physical balance scales representing the notion of balance, and weights representing numbers. While past research has indicated that the balance model has its limitations (Aczel, 1998), it has its advantages in that it considers both the right hand and left hand sides of equations and is not directional in any way (Pirie & Martin, 1997). It also copes with the need to attend to the equation as an entity rather than an instruction to achieve a result. Its limitation lies in its inability to model subtraction equations or unknowns as negative quantities, and thus is not seen as fostering a frame of mind adequate for the full development of algebraic thinking (Aczel, 1998).

Concrete models are, however, endowed with two fundamental components (Filloy & Sutherland, 1996), namely, *translation* and *abstraction*. Translation encompasses moving from the state of things at a concrete level to the state of things at a more abstract level. The model acts as an analogue for the more abstract. Abstraction is believed to begin with exploration and use of processes or operations performed on lower level mathematical constructs (English & Sharry, 1996; Sfard, 1991). Through expressing the nature of their experience and articulating their defining properties, learners can construct more cohesive sets of operations that subsequently become expressions of generality. It is conjectured that abstraction entails recognising the important relational correspondence between the balance model, the weights as numbers, stating the identity of the two quantities or expressions (i.e., the two sides of the balance scales), and applying this understanding to more abstract situations, such as, equations involving subtraction and unknowns as negative numbers. Filloy and Sutherland (1996) suggest that not only do models often hide what is meant to be taught, but also present problems when abstraction from the model is assigned to the pupil. Teacher intervention is believed to be a necessity if the development of detachment from the model to construction of a new abstract notion is to ensue.

The problems that the students were asked to solve were simple addition and subtraction problems with one unknown. The specific aims of this research were to ascertain if students:

- Comprehended the language used to describe equations and inequations;
- Recognised equals as quantitative sameness;

- Translated the balance model as a model for equations; and
- Utilised the balance model for solving for unknowns.

Methodology

The methodology adopted was that of a Teaching Experiment, the conjecture-driven approach of Confrey and Lachance (2000). The experiment was exploratory in nature and consisted of three phases, namely, developmental, experimental, and implementation. In the developmental phase two researchers taught two classes for one week. The teaching episodes focused on developing early understandings of equal as balance and extending this concept to using the balance strategy to solve for unknowns in simple addition and subtraction equations, with an emphasis on appropriate language, models and symbols. During and in between each lesson hypotheses were conceived "on the fly" (Steffe & Thompson, 2000) and were responsive to the teacher-researcher and the students. In the experimental phase two new teachers, with the assistance of the researchers, taught the sequence of lessons in their classrooms and appropriate modifications to the lessons were made. Finally, the modified lessons were taught in nine other classrooms by nine other teachers, without any assistance from the researchers. Between each phase, all of the teachers and the two researchers met to discuss the lessons, the underpinning concepts and understandings, and any difficulties that students were experiencing.

The lessons consisted of four main dimensions, namely: (a) developing language that illustrated equivalence and non-equivalence situations; (b) establishing equals as more than a syntactic indicator or didactic sign; (c) translating the physical balance model to symbolic equations; and (d) using the balance strategy to evaluate the unknown in simple addition and subtraction equations. The teaching episodes occurred over a four lesson sequence, with each lesson taking approximately one hour. In each lesson the four dimensions discussed above were the primary focus.

The language chosen for representing *equal* and *not equal* situations in the teaching phase were: *same [value] as, equal, balanced, different [value] from, not equal,* and *unbalanced*. The symbols used were "=", "≠", and equations represented horizontally, with more than one value following the equal sign. The concrete model used to represent these ideas comprised balance scales with small cans of baked beans and of spaghetti to represent numbers. The cans were of the same mass and shape. It was conjectured that the two different types of cans allowed students to "see" 2 + 3 as a concept rather than as a process of finding the answer "5", that is, seeing the expression rather than the answer, an important dimension of algebraic thinking (Tall, 1991). Figure 1 illustrates the pictorial model used for addition situations.

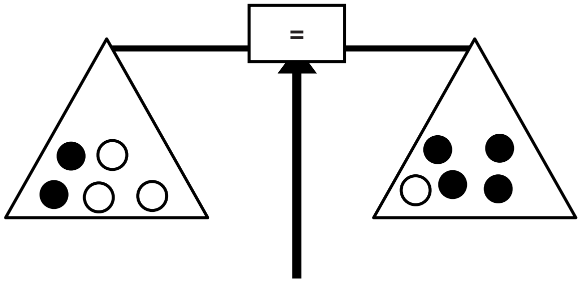

Figure 1. Pictorial model for balance illustrating 2 + 3 = 4 + 1.

During the teaching phase all three models were represented, that is, the concrete model, the pictorial model, and the symbolic model, with students being required to continually translate between the models (e.g., represent 2 + 3 = 4 + 1 with the concrete model or the pictorial model) and express each with appropriate language (e.g., 2 plus 3 is the same value as 4 plus 1, or 2 plus 3 equals 4 plus 1). Equivalence was explained as occurring when the scales were balanced.

The sequence of lessons proceeded from establishing the notions of *equals* and *not equals*, examining equals as quantitative sameness in arithmetic situations, exploring the operation that could be applied to both sides of an equation to keep it balanced, to utilising this strategy to find one unknown in simple addition and subtraction equations. For the physical balance, the unknown was represented by a "mystery" bag with an unknown number of cans of baked beans in it. During the teaching phase students were asked to offer representations for the unknown in symbolic form. After some discussion the symbol was chosen – a question mark within a small square. This symbol was a student-invented symbol. Given the age of the students, it was decided to use this symbol rather than the more formal symbol x.

During the lessons, the teacher-researcher continuously asked three types of questions: *reflecting* (e.g., What did you do? What happened? Why did this happen?), *predicting* (e.g., Before you put that on, what do you think will happen? Will they be equal?), and *modifying* (e.g., Is this correct?). At the conclusion of the lessons in both the developmental and experimental phases, the researchers and teachers reflected on the learning that occurred and made the necessary adjustments to the lesson sequence and materials used. These modifications reflected teacher feedback and student responses. Both the developmental and experimental classes were videotaped.

Sample

Thirteen Year 3 classes from medium to high socio-economic areas of a major Australian city participated in the project. The average age of the students was 8 years. In order to ascertain students' understanding of the concepts delineated in the lessons, a random sample of 20 students was chosen from four classes, the two developmental and the two experimental classes, for in-depth one-on-one interviews. The aim of the interview was to assess the robustness of the ideas developed in the teaching experiment and investigate how students used these ideas to solve for the unknown in simple arithmetic equations. These interviews were of approximately 40 minutes duration and each interview was audio-taped.

Instrument

The interview consisted of four tasks. Each task reflected the four dimensions of the teaching experiment, namely, language for describing equivalent and non-equivalent situations, equals as quantitative sameness, translating the physical model to symbolic equations, and balance as a strategy for finding unknown quantities. While the tasks chosen for the interview were different from those used in the teaching experiment, they reflected the ideas explored in the classroom context.

Language to describe equivalent and non-equivalent situations

The language task required students to read each card, sort cards into groups, give a name to each group, explain why particular cards were placed with each other, and describe how these words would be used in mathematics. This approach was based on the work of Luria (1981) of defining, comparing, differentiating and classifying words, providing a "powerful instrument for assessing students' understanding and ample opportunity to explore the multiple meanings children have for words" (Berenson, 1997, p.2). The words used for this task are shown in Figure 2. These words reflected the language and symbols utilised throughout the lesson sequence.

Different from	Same as	Equals
Not equals	Balanced	Unbalanced
Equation	=	≠

Figure 2. Cards used in Language Task (Task 1).

Equals as quantitative sameness

For this task students were presented with the cards in Figure 3 and asked if they were true or not true and to explain their answers.

$6 = 2 + 4$	$2 + 3 = 6 - 1$

Figure 3. Cards used for the Equations in Different Formats task.

These cards were designed to ascertain if students recognised an equation with more than one number after the equal sign, that is, if they viewed = as more than an indicator to place the answer (Carpenter & Levi, 2000).

Translating the physical model to symbolic equations

In this task the interviewer modelled 3 + 4 = 7 and $\boxed{?}$ + 7 = 11 with the balance scales and cans, and pointing to the model asked each student, "What do you think this means?" The unknown was represented by a bag with question marks on it and four cans were placed "secretly" in the bag by the interviewer.

Use of balance to solve for unknowns

For the fourth task, students were presented with four cards, one at a time, and were asked to solve for the unknown and explain how they arrived at their answers. Figure 4 presents the questions asked on each card.

1	?	− 4 = 13

2		?	+ 7 = 11

3	5 +	?	= 12

3	?	+	?	+ 2 =	?	+ 5

Figure 4. Four cards used in the interviews.

The number on each card indicates the position in the sequence in which the cards were given to the students. Card 4 did not form part of the lessons and was included to determine if any students had abstracted the process of solving with balance and could use it to solve for equations with unknowns on each sides of the equal sign. Given the age of the students, we decided to express the fourth card (2? + 2 = ? + 5) in the format ? + ? + 2 = ? + 5, where ? represented the unknown. Before the interview commenced, it was determined that, as students began to struggle with their responses, the interviewer would discontinue asking questions for that particular task. At a later stage these students would be withdrawn for one-on-one instruction.

Results

Language

Responses to the language task indicated that 18 students sorted their cards into two predominant groups. The first group consisted of *equals, same as*, =, and *balanced* and the second group was *not equals*, ≠, *unbalanced*, and *different from*. Nine of these students were unsure what equation meant, they simply left it out of the sorting activity. The groups tended to be named balanced or unbalanced. The explanations given for why the cards belonged together varied but most gave aspects of Elizabeth's (age 8) explanation, which was:

> If you have 5 baked bean cans on one side and 5 baked beans on the other that is the same as… It equals the same amount it is balanced… Unbalanced means if you have 3 in one and 4 in the other it is not the same… it is different from… It is a different number… It is unbalanced.

The responses from other students differed in terms of the examples they chose to represent the groups, and some did not include specific reference to the cans but simply talked in terms of numbers. One student created five groups, two as above and three with one card in each which were: equation, different from, and same as. Another student created two extra groups, one with equation and one with different from.

Equals as quantitative sameness

All 20 students stated that 6 = 4 + 2 was true. The predominant explanation (15 students) involved restating the problem as "true because 4 plus 2 equals 6". Four of these students included statements such as "6 is 4 plus 2". Only three students simply said it was true "because 6 equals 4 plus 2", and two students said it was true "because it equals". Nineteen students believed that 3 + 2 = 6 – 1 was true, and 18 of those students stated it was true because "3 plus 2 equals 5 and 6 minus 1 equals 5". Only one student believed that the statement was not true "because of the minus 1, 3 plus 2 is true but 6 minus 1 is not true... you can't have minus 1 on this side".

Representing equivalent situations with materials

All correctly identified the model of 3 plus 4 equals 7 with a common response being "7 is three plus 4…7 equals 7", referring to the balance scales. They could also recognise the model of the second representation. When asked to find the unknown, 10 students simply stated that it's 4 "because 4 and 7 is eleven" and the other 10 students found the unknown by physically taking four cans from each side.

Using a balance to solve for unknowns

With regard to the fourth task, students used a variety of methods to solve for the unknown. Tables 1, 2 and 3 summarise their responses to the first three cards presented in this task.

Table 1
Frequency of Responses for ? – 4 = 13

Category	**Description**	**Frequency of response**	**Frequency of correct response**
Balance	Add 4 to both sides	10	10
Guess and check	16 take 4 is 12… 17	4	3
Ignored unknown	4 from 13 is 9	3	0
Partial response	Count from something to get 13	2	2
No response		1	0

Each category shown in Table 1 expresses the strategies used by these students in response to the question within the interview. Thirteen students correctly responded to this task. The most common strategy used to solve for the unknown was the balance method, adding 4 to both sides. Of interest were the three students who simply combined all the signs presented in the equation (13, 4 and –) and gave the answer 9. This was not a strategy they used for ? + 7 = 11 (i.e., 7, 11 and + is 18).

Table 2
Frequency of Responses for ? + 7 = 11

Category	**Description**	**Frequency of response**	**Frequency of correct response**
Balance	Take away 7 from both sides	8	8
Guess and check	2 no 9, 3 no 10, …	3	3
Number Fact	4 + 7 = 11 because 3 + 7 = 10	6	6
Counting on	8, 9, 10, 11... it's 4	2	2
Partial response		1	0

Nineteen students correctly responded to the task *? + 7 = 11*. Again, as shown in Table 2, the most common strategy used to solve for the unknown was the balance method, subtracting 7 from each side. Unlike the first example, six students simply used number facts to reach solutions.

Table 3
Frequency of Responses for 5 + ? = 12

Category	**Description**	**Frequency of response**	**Frequency of correct response**
Balance	Take away 5 … 12 take 5 …7	7	7
Guess and Check	5 no, 6 no, 7	1	1
Number facts	5 + 7 is 12	3	3
Counting on	6, 7, 8, 9, 10, 11, 12	1	1
Unsure	Unknown should be first	1	0
No response		7	0

Twelve students offered correct responses for the task 5 + ? = 12. Seven students were perceived by the interviewer to struggle with finding the unknown and explaining how they reached solutions. For two students, the difficulty appeared to result from the placement of the unknown in the second position rather than in the first position (i.e,. ? + 5 = 12). Comments included "It is wrong", and "I can't do it, not there". In both cases these students pointed to the ?. Seven students could not respond to the task and, at this stage of the interview, they were ascertained to need further teaching instruction.

The final task in this phase (? + ? + 2 = ? + 5) was included to see if any students were capable of applying the balance strategy to solve more complex problems. This type of problem did not form part of the teaching sequence. Three students used a *guess and check* method, but in each instance they assigned different values to each of ?. Only three students were able to use the balance method to find the unknown. The following extracts from the transcripts delineate how the students reached solutions for this problem.

Guess and check using different numbers.

> We have 2 mystery bags here plus another 2 and the mystery bag and 5. It is a little hard. It could be 2 there [the first ?] and so it would be 6 [the second ?] and in this mystery bag you would have 1 [the third ?]. No.
>
> There is 10 in this mystery bag [the third?] and there is 9 in this mystery bag [the first ?] and there is 1 [the second?] in this one. (9 + 1 + 2 = 10 + 5) No 6 No 4. Because 9 plus 4 equals No it has to be 3. So 9 + 3 + 2 equals 10 + 5.

Used balance to find the unknown

> That's hard I will use the balance scales. There must be one baked bean in the mystery bag and one baked bean in the other bag and that equals 2 No plus 2 so that has to be… we don't know what these are just yet. If I took away 2 from 5 that would give me 3 [? + ? = ? + 3] so there is 3 in one of these mystery bags. I think there is 3 in each mystery bag.
>
> This is hard. Can I move the side around? I could do 5 plus 2 equals 7. So 7 plus 2 plus 5 and then found out what that answer is and that could be the one of these. So that could be 2 plus 2 plus 2.
>
> *[I. Yes but we have to get it to balance. There is a way to do it to find out what there are].*
>
> Take away 2. And put a take away 2 at the end. So 5 take away 2 equals 3 so one of those question marks might be 3. 5 take away 2 equals 3... 5 plus 3 equals 8 so 3 plus 3 plus 2 equals 8.

In the two latter cases the students only used the balance to remove the 2, giving ? + ? = ? + 3. From this pattern they seemed to simply see that each ? must be 3 in order for the equation to be correct. Table 4 summarises the frequency of responses for this task.

Table 4
Frequency of Responses for ? + ? + 2 = ? + 5

Category	**Description**	**Frequency of response**	**Frequency of correct response**
Balance	Take 2 from both sides… 3	3	3
Guess and Check	Different numbers for each unknown	3	0
Number facts	1 + 1 + 2 + 1 = 5 so ? must be 1	1	0
Grouped knowns	5 + 2 = 7 and so one is 7	1	0
No response		5	0
Not asked		7	0

The other responses to the task ? + ? + 2 = ? + 5 included one student who ignored the = and assigned 1 to each of the unknowns; another student simply ignored the unknown and combined 5 and 2 to reach 7 as the value for one unknown.

Discussion and Conclusions

The aim of these interviews was to ascertain students' interpretations of the balance model, particularly its application to solving for unknowns. The card sorting activity indicated that most students did not experience difficulties in understanding the language introduced in this lesson sequence to describe the differences between equivalence and inequivalence, or the use of the balance model to represent these situations. Twenty of them grouped balance with equals, and unbalanced with not equals. Many were unsure of equation. Given that equation was only briefly referred to in the lesson sequence, this is not surprising. Two students failed to group different from with unbalanced and unequal. Their explanations indicated that they were relating the words to the balance scale, "8 is different from 1": "The heavier side would go down so it wouldn't be equal", and "It would be different from", refuting our original conjecture that the confusion may be occurring because of the use of difference in subtraction situations, but still they did not want to include it in the unbalanced group.

All students recognised 6 = 4 + 2 as an appropriate representation for an equation, with only three stating that "6 equals 4 plus 2". Fifteen students reversed the order of the equation, suggesting that even though they recognised it as true, they still tended to privilege an interpretation where the equal sign means that they should carry out the calculation that proceeds it, and the number following the equation sign is the answer to the calculation (Saenz-Ludlow & Walgamuth, 1998). Four of these students also stated that "6 is 4 plus 2", giving both responses, with three giving this response before reversing the equation. Of interest were their responses to 3 + 4 = 6 – 1. Nineteen believed it was true and they justified their answers by stating the identity between the two expressions. This trend goes against findings from past research with similar aged children (Carpenter & Levi, 2000; Warren, 2001) and supports the conjecture that explicit teaching and use of appropriate models and language may begin to assist young children move beyond interpreting equals as a place for the answer. The balance model did appear to move the focus of student attention away from = indicating an answer, towards attending to an equation as an entity (Pirie & Martin, 1997). All students also recognised the physical balance model and successfully translated from the model to appropriate language and symbolic notation, indicating that the explicit teaching of the model was effective (Filloy & Sutherland, 1996).

The number of correct responses for finding the unknown varied; 19 gave the correct answer for ? + 7 = 11; 13 gave the correct answer for ? – 4 = 13; and 12 gave the correct answer for 5 + ? = 12. This would suggest that these children found it easier to solve addition problems with the unknown in the first position than either subtraction problems or addition problems with the unknown in the second position. The strategies utilised for each also varied. The most common strategy used was a balance strategy incorporating the isolation of the unknown by applying the inverse operation, followed by the use of a numerical strategy (i.e., counting on; counting down; number facts) to reach a solution. Seven children consistently used this strategy across the three tasks.

Interestingly, not one child used number facts to solve for the unknown in the subtraction problem. This contrasts with the six who found the unknown for ? + 7 = 11 by saying "it is 4 because 4 plus 7 equals 11". For the subtraction problem, 10 children used the balance strategy. As this type of problem cannot be modelled with concrete materials, it is surmised that by unearthing the underlying mathematical features of the balance model and by applying it to more abstract problems, these children had moved beyond the limitations of the balance model (Azcel, 1988). This inference is also supported by the finding that three children successfully used the strategy to find the unknown in task 4 (? + ? + 2 = ? + 5). They had begun to transfer the process to more complex problems, however, they only used the strategy to eliminate the known from one side of the equation (take 2 from both sides) and did not use the strategy to take an unknown from each side. This could reflect the structure of the question, and might suggest that once they eliminated the known, they were left with ? + ? = ? + 3, a structure from which they easily recognised that the unknown must be three. The difficulty that children experienced with this equation could also reflect the conjectured increase in complexity from Card 3 to Card 4. Boulton-Lewis et al. (1998) suggest that the prerequisite knowledge required for success at this level involves success at binary arithmetic operations (2 + 3 = 5), complex arithmetic (a series of operations on numbers), and binary algebraic operations ($x + 3 = 11$).

The equation ? + ? + 2 = ? + 5 represents an example of what Filloy and Rojano (1989) refer to as the *didactic cut* between arithmetic and algebra, a first-degree equation with unknowns on both sides. They suggest additional resources are needed to overcome this barrier. It seems that the introduction of the balance model begins to assist children bridge this gap at an early age. While we did emphasise in the teaching experiment that in these types of equations ? does represent the same amount, three children assigned different numbers to each unknown. This could reflect the types of representations previously used for unknowns. Commonly many texts provide examples such as □ + □ = 10, and children are asked to write all of the ways they can find 10. In these texts the same symbol is deliberately assigned different values, whereas in algebra the same symbol always has the same value. The impact this practice has on later algebraic experiences needs further investigation.

Of interest were the other strategies used by students to reach solutions. From a comparison between the solutions used for cards 1 and 2, it appeared that some students simply combined the signs for the subtraction problem (i.e., 13 – 4 = 9 indicating that 9 was the unknown), but did not apply the same strategy to the addition problem (i.e., 7 + 11 = 18). Another interpretation of their error is that in both problems, the students simply subtracted the smaller number from the larger number to find the unknown. This strategy is consistently valid for addition problems with one unknown but not for subtraction problems with one unknown. For instance, 7, 4 and 11 form four triads used commonly to explore the relationship between addition, subtraction and number facts (i.e., 7 + 4 = 11; 4 + 7 = 11, 11 – 7 = 4; 11 – 4 = 7). The only possible equations for 4, 11, + and ?, with the unknown are (i) ? + 4 = 11, and (ii) 4 + ? = 11. In both instances the position of the unknown does not affect the solution. Thus when children see +, 11, 4 in an equation they instantly think of 7, a generalisation that is valid. This generalisation does not apply for the problem presented on card 1. The position of the unknown does affect the triad chosen. For example, ? – 4 = 13 requires 17, 4 and 13 whereas 13 – ? = 4 requires 13, 9 and 4. In both instances the equations are composed of –, 13, and 4. Thus the strategy that consistently works for addition situations does not always work for subtraction situations. It is surmised that these children had formulated a generalisation for addition problems with one unknown and falsely applied it to subtraction problems. This supposition requires further investigation.

The researchers acknowledge that one perceived limitation to this research is the absence of a pre-test, but in on the fly (Steffe & Thompson, 2000) teaching experiments this presents difficulties. The comparison between these students' responses and similar aged students from past research (e.g., Carpenter & Levi, 2000; Warren, 2001, 2002, 2003), suggests that the use of the balance model assisted some children in approaching problems with unknowns, particularly solving problems with unknowns on both sides of the equal sign. It also appeared to provide a language base that assisted children in explaining their solutions. In addition, it seemed to shift the focus away from = as indicating an answer towards = as equivalence. Given that the teaching experiment only occurred over a four-lesson sequence, this finding is encouraging. While the model assisted many children, additional focussed teaching was necessary for others. The extent and type of intervention required is still under investigation.

The results from this research give clear directions for further research in this important area of algebraic thinking. Not only were students beginning to exhibit characteristics of Freudenthal's (1983) third category of understanding the equal sign (a statement that something is true for all values of the variable), but from their responses to Task 4, a strong direction for further teaching was signalled. Three students successfully used the balance strategy to find the unknown and were clearly demonstrating early algebraic thinking that could be promoted through further planned instruction following from the initial instruction presented in this phase of the research.

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